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# Numerical solution of the one-dimensional saddle point equation of the Ginzburg–Landau Hamiltonian with random temperature

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## Abstract

The one-dimensional saddle point equation of the Ginzburg–Landau Hamiltonian with random temperature is studied with a numerical method. The random temperature is correlated with a finite range  $l$ . The distribution width of the random temperature is  $\Delta$ . The ground state of the saddle point equation is solved. The average, fluctuation and auto-correlation of the order parameter are obtained. It is found that the auto-correlation function behaves like  $\sim \exp(-x^2/\xi_\phi^2)$ . For  $\Delta \gg 1/l^2$ , where  $l$  is dimensionless, the correlation length is given by  $\xi_\phi \propto l$ . For  $\Delta < 1/l^2$ , as  $t = 0$ , the correlation length  $\xi_\phi \propto l\Delta^{-\alpha}$ , where  $\alpha = 0.65$ . All the saddle point solutions for  $\Delta < 1/l^2$  can be mapped to that for  $\Delta = 1/l^2$  by using a coarse-grained approximation.

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## 1. Introduction

Recently the interest in the saddle point equation of the Ginzburg–Landau Hamiltonian with random temperature has been stimulated in discussing the critical phenomena in quenched disordered systems [1–4]. Conventional renormalization group considerations assume that the order parameter is zero above the critical temperature [5–7]. However, it is pointed out that there may exist many local minima solutions for the disordered Hamiltonian above the critical temperature [1–4].

Physically this situation is quite clear: owing to the spatial fluctuation of the local transition temperature one can find a macroscopic number of ‘ferromagnetic islands’ above the critical temperature. As the temperature falls, these islands get connected and form larger clusters. As these islands percolate, a phase transition takes place.

This idea was discussed very early in the context of dirty superconductors. Bulaevskii *et al* have investigated a very dirty superconductor near the Anderson localization threshold using the replica method [8]. They mainly discussed the stability of the isolated islands of

the ordered phase. Pentegov *et al* discussed the effect of the localized modes on the tricritical phenomena in disordered systems [9]. They expand the order parameter in the eigenfunctions of the Schrödinger equation with random potential. The droplet state they discussed has high energy, and hence it is strongly localized. However, these instanton solutions are valid only if the islands are far away from each other. That is, the system is far from the critical regime.

Dotsenko *et al* proposed a replica symmetry breaking scheme in the renormalization group approach to take into account these local minima of the disordered Hamiltonian [1, 2]. It was eventually found that the renormalization flow leads to the strong coupling regime and not to the expected fixed points. Considering this strange result, Tarjus and Dotsenko studied the locally ordered regions by solving the saddle point equations of the replicated Hamiltonian within the framework of the Gaussian variational approximation [4]. They proposed that there may exist a spin-glass phase in the random temperature Ising ferromagnet.

Korzhenevskii *et al* proposed a finite-range correlated disorder model to study the percolation of the locally ordered regions [3]. They gave a criterion for the ordered regions; the region is locally ordered if  $\xi(\mathbf{r}) = a|t(\mathbf{r})|^{-1/2} < l$ , where  $\xi(\mathbf{r})$ ,  $t(\mathbf{r})$  and  $l$  are the thermal correlation length, local reduced temperature and the size of the region, respectively, and  $a$  is a constant. The constant  $a$  can be set to be 1 and in the following discussion we adopt  $a = 1$ . The argument for this criterion is that if  $\xi < l$ , a spontaneous thermal fluctuation of range  $\xi$  is not sufficient to flip the state of the region, so it can sustain its local order. They consider a lattice of cells of size  $l$ , the distribution width of the random temperature in each cell being  $\Delta$ . According to their criterion, for  $\Delta \gg 1/l^2$ , the systems will be locally ordered first and then the locally ordered regions percolate as the temperature decreases. In this case the phase transition is percolative. The percolation of the locally ordered regions is the usual site percolation, where a site is occupied randomly with a certain probability, independent of its neighbours [10]. For  $\Delta \ll 1/l^2$ , there exists no locally ordered regions and the phase transition is ‘homogeneous’, which means that the systems become ordered globally as the temperature decreases to the critical temperature.

Recently experiments also show the existence of locally ordered regions above the critical temperature. For example, it is indeed found that there exists localized Bose–Einstein condensation above the macroscopic superfluid transition temperature [11]. On the other hand, the superfluid transition of  $^4\text{He}$  in porous media is difficult to understand by the conventional theory of critical phenomena in disordered systems [12, 13].

Although the physics of the locally ordered regions has been discussed by many authors, we lack detailed knowledge of the locally ordered regions, such as the size, the amplitude of order parameter and their dependence on the strength of the disorder and the temperature, etc. In this paper we discuss the locally ordered regions by directly solving the saddle point equation with random temperature.

We consider a lattice of cells of size  $l$ , the distribution width of the random temperature in each cell being  $\Delta$ . It is a practical model. As we know, the Ginzburg–Landau Hamiltonian with random temperature is studied in order to understand the critical behaviour of the diluted spin model [6]. From the diluted spin model, Grinstein and Luther derived a replicated Hamiltonian. We have shown that it can be mapped to the finite-correlated disorder model with the cell size  $l$  being just the spin lattice spacing, and the disorder strength  $\Delta = \sqrt{2p(1-p)}$ , where  $p$  is the spin occupation probability. Therefore, for the diluted spin model it is satisfied that  $\Delta < 1/l^2$ .

In this paper the one-dimensional saddle point equation on the lattice described above is solved. As the first step we solve the saddle point equation for a single well surrounded by barriers. (For convenience, we call the cells with negative temperature wells, and the cells with positive temperature barriers.) We find the dependence of the amplitude of the solution on the well depth and the barrier height. It is shown that Korzhenevskii’s criterion [3] for

the locally ordered cells is too simplified. Although the threshold value of the well depth is indeed approximately proportional to the inverse of the square of the well size, Korzhenevskii's criterion ignores the effects of the configuration of the cells. Whether the saddle point solution in a well is zero or not depends on not only itself but also the surroundings. For different surroundings the criterion is different. Therefore, the percolation of the locally ordered regions should not be the usual site percolation with Korzhenevskii's criterion.

Then we study the one-dimensional saddle point equation for the disorder strength varying from  $\Delta = 100/l^2$  to  $\Delta = 0.01/l^2$ . The ground state is obtained by a numerical method. The average and the fluctuation, and the autocorrelation function of the order parameter for the ground state are calculated. The correlation length is found to be the averaged size of the localized solution.

For  $\Delta \gg 1/l^2$ , which is called strongly disordered, the dependence of the average and fluctuation on the temperature has the same form approximately. The correlation length is around the cell size. These properties can be understood by the result for a single well. The saddle point solutions are well localized in single wells. The size of locally ordered regions is about the cell size. Therefore the correlation length of the solution is around the cell size.

For weakly disordered systems, where  $\Delta \ll 1/l^2$ , the consideration of single wells is misleading. According to Korzhenevskii's criterion [3] there should be no locally ordered regions in this case, but in our numerical calculation it is found that for  $\Delta \ll 1/l^2$  there still exist localized solutions. However, the solutions are not localized within single wells, but extend over many cells. It is shown that the single well consideration is not suitable for the solution on a whole lattice for  $\Delta \ll 1/l^2$ .

For  $\Delta \ll 1/l^2$ , it is found that for  $t = 0$ , the correlation length  $\xi_\phi \propto l\Delta^{-\alpha}$ , where  $\alpha = 0.65$ . Considering that the ordered regions extend over many cells, we propose a coarse-grained approximation to explain the numerical results. We argue that  $\xi \propto l\Delta^{-2/3}$ . All the saddle point solutions for  $\Delta < 1/l^2$  can be mapped to that for  $\Delta = 1/l^2$  by using the coarse-grained approximation. The generalization of the coarse-grained approximation to higher dimensional models is also given.

To avoid misunderstanding, it should be emphasized that we are not studying the critical phenomena in one-dimensional systems, although the one-dimensional saddle point equation is solved in this paper. In fact the one-dimensional Ginzburg–Landau Hamiltonian is not an effective Hamiltonian of one-dimensional spin models, such as the Ising model. Indeed the true behaviour in one dimension is not described by the Ginzburg–Landau Hamiltonian. For example, the system orders only for  $T = 0$  and therefore fluctuations in reduced temperature can only be positive and not negative. In addition the correlation length diverges exponentially as  $T = 0$  is approached and not like the inverse square root of the reduced temperature. We are interested in the critical phenomena in spin systems with dimension larger than one. Only the Ginzburg–Landau Hamiltonian with dimension higher than one can be related to the critical behaviour of spin models. The one-dimensional saddle point equation, which is studied in this paper, is only a simplified modelization of the high-dimensional ones. In this equation the local critical temperature is finite, and the fluctuation in reduced temperature can be both positive and negative. We expect that the solution of the one-dimensional model can provide insight into the high-dimensional ones. In addition, the calculation in one dimension can have high precision because one can use large lattices in one dimension, where the boundary effect, the deviations from the average among different samples can be small. At the end of the paper one can see that the coarse-grained approximation indeed gives a useful clue to investigate the high-dimensional saddle point equations.

In the next section, we discuss some useful properties of the Ginzburg–Landau Hamiltonian with random temperature and the one-dimensional model. In section 3, we solve the saddle point equation with a single well surrounded by barriers. In section 4, we report our numerical result for the saddle point equation on one-dimensional lattices. In section 5, we propose the coarse-grained approximation to explain the numerical results for  $\Delta < 1/l^2$ . Section 6 is a summary. In appendix A, we show the mapping of the diluted spin model to the Ginzburg–Landau model with random temperature and discuss the correlation of the disorder and the strength of disorder. In appendix B, the details of the numerical method are given.

## 2. The model

For  $d$ -dimensional systems, the Ginzburg–Landau Hamiltonian with random temperature reads

$$H = a^{-d} \int d\mathbf{x} \left( \frac{1}{2} a^2 |\nabla \phi|^2 + \frac{1}{2} (t + \tilde{t}(\mathbf{x})) \phi^2(\mathbf{x}) + \frac{1}{4} g \phi^4(\mathbf{x}) \right) \quad (1)$$

where  $t$  and  $\tilde{t}(\mathbf{x})$  are the average temperature and random part caused by the disorder, respectively. Generally  $t + \tilde{t}(x) = (T - T_c(\mathbf{x}))/T_c(\mathbf{x})$ ,  $T, T_c(\mathbf{x})$  are the temperature of the system and the local critical temperature, respectively. Here we assume that the local critical temperature  $T_c(\mathbf{x})$  is finite.  $\phi$  is a  $m$ -component order parameter.  $a$  is the lattice spacing, which can be set to be the unit of length. In this paper we only consider the Ising case  $m = 1$ . It can be shown that the ground states of the saddle point solution for  $m = 1$  and  $m > 1$  are the same. The difference between these two cases is that the excited states for  $m = 1$  have domain walls while those for  $m > 1$  have Goldstone modes. Here we do not consider excited states.

In our model we consider a lattice of cells and assume each cell has a local temperature

$$\tilde{t}(\mathbf{x}) = \tilde{t}_i \quad \mathbf{x} \in i\text{th cell} \quad (2)$$

where  $i = 0, 1, 2, \dots$ , and the cell size is  $l$ . The probability of  $\tilde{t}_i$  is given by

$$p(\{\tilde{t}_i\}) = \frac{1}{\Delta \sqrt{\pi}} \exp\left(-\frac{\tilde{t}_i^2}{\Delta^2}\right). \quad (3)$$

Through simple calculation, we get the correlation of the random temperature

$$c_t(\mathbf{x}, \mathbf{y}) = \langle \tilde{t}(\mathbf{x}) \tilde{t}(\mathbf{y}) \rangle_{\text{av}} = \begin{cases} \Delta^2/2 & \mathbf{x}, \mathbf{y} \in \text{the same cell} \\ 0 & \mathbf{x}, \mathbf{y} \notin \text{the same cell} \end{cases} \quad (4)$$

where  $\langle \dots \rangle_{\text{av}}$  is the average over the random temperature. Obviously the correlation length of the disorder is the cell size  $l$ .

An important system with finite range correlated disorder is the diluted spin model. In appendix A, we show that the correlation length of the disorder is the lattice spacing, and the strength of the disorder is  $\Delta = \sqrt{2p(1-p)}$ , where  $p$  is spin occupation probability.

Another interesting system with finite range correlated disorder is  $^4\text{He}$  in porous media. In the experiment of superfluid transition of  $^4\text{He}$  in porous media, the disorder caused by the media is long but finite range correlated [12, 13].

The saddle point equation is given by

$$-\nabla^2 \phi(\mathbf{x}) + (t + \tilde{t}(\mathbf{x})) \phi(\mathbf{x}) + g \phi^3(\mathbf{x}) = 0. \quad (5)$$

Here the constant  $a$  in equation (1) is set to be 1. There are three parameters: the cell size  $l$ , the strength of disorder  $\Delta$  and the coupling constant  $g$ .

We introduce the following transformations

$$\mathbf{x}' = \mathbf{x}/b \quad t'(\mathbf{x}) = b^2 t(\mathbf{x}) \quad \phi'(\mathbf{x}) = b\sqrt{g}\phi(\mathbf{x}) \tag{6}$$

then we have

$$-\nabla'^2 \phi' + (t' + \tilde{t}'(\mathbf{x}'))\phi'(\mathbf{x}') + \phi'^3(\mathbf{x}') = 0. \tag{7}$$

The probability distribution of  $\tilde{t}'$  becomes

$$p(\{\tilde{t}'_i\}) = \frac{1}{\Delta'\sqrt{\pi}} \exp\left(-\frac{\tilde{t}'_i{}^2}{\Delta'^2}\right) \tag{8}$$

where  $\Delta' = b^2\Delta$ . Therefore, one can solve the saddle point equation with  $l = g = 1$ . Through the solution of equation (7) and the transformations (6)–(8) one can get the solution of equation with arbitrary  $l, g$ . So there is only one concerned parameter  $\Delta'$ . Throughout this paper we set  $l = 1, g = 1$  in the numerical calculation.

There are two characteristic length scales in the above model. One is the cell size  $l$ , another is  $1/\sqrt{\Delta}$ , which is the local thermal correlation length as the local temperature is equal to  $\Delta$ . As the average temperature is lowered to  $t \sim \Delta$ , the probability of wells is not small. The percolation of ordered regions takes place in this regime. Our calculation is focused on this regime.

In the next two sections, we solve the one-dimensional saddle point equation. It should be noted that the one-dimensional saddle point equation studied here is only a simplified modelization of the high-dimensional ones. In this equation the local critical temperature is finite, and the fluctuation in reduced temperature can be both positive and negative. It is expected that the solution of the one-dimensional equation can provide insight into the high-dimensional ones. We are not studying the critical phenomena in one-dimensional systems, such as the Ising model, which orders only for  $T = 0$ . The true critical behaviour in one dimension is not described by the above Ginzburg–Landau Hamiltonian.

### 3. The solution of a single well

To understand the solution on a whole lattice, we consider a single well case at first. The simplest one-dimensional case is a well with depth  $-t_w$  between infinitely high barriers and is written as

$$-\frac{d^2\phi(x)}{dx^2} + t(x)\phi(x) + \phi^3(x) = 0 \tag{9}$$

and

$$t(x) = \begin{cases} -t_w & |x| < l/2 \\ \infty & |x| > l/2 \end{cases} \tag{10}$$

where  $t_w > 0$ . Obviously the solution must satisfy

$$\phi(x)|_{x=\pm l/2} = 0. \tag{11}$$

There is a detailed discussion about the above equation in [14]. In regions with constant local temperature, one finds immediately a first integral of

$$\alpha = 2\left(\frac{d\phi}{dx}\right)^2 + 2t_w\phi^2 - \phi^4. \tag{12}$$

For  $0 < \alpha < t_w^2$ , there are two kinds of solutions. One is a sine-like function with period

$$\lambda = 4 \int_0^{\phi_{\max}} d\phi \left(\frac{1}{2}\phi^4 - t_w\phi^2 + \frac{\alpha}{2}\right)^{-1/2} \tag{13}$$

and extremes

$$\phi^2 \leq \phi_{\max}^2 \leq t_w - \sqrt{t_w^2 - \alpha}. \quad (14)$$

The solution with half period equal to  $l$  can match the boundary condition. For certain  $t_w$  the shortest period is given by

$$\alpha \rightarrow 0 \quad \lambda \rightarrow \lambda_{\min} = \frac{2\pi}{\sqrt{t_w}} \quad (15)$$

and also  $\phi \rightarrow 0$  as  $\alpha \rightarrow 0$ . Therefore if

$$l < \lambda_{\min}/2 = \frac{\pi}{\sqrt{t_w}} \quad (16)$$

there exists no nonzero solution for equation (9).

We give a simple variational method to discuss the meaning of the criterion (equation (16)) and show some important features clearly. To satisfy the boundary condition we use the following form of  $\phi$

$$\phi = \phi_0 \cos \frac{\pi}{l} x \quad (17)$$

as the trial solution. Substituting it into the Hamiltonian we obtain

$$\begin{aligned} H &= \int_{-l/2}^{l/2} \left( \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 - \frac{1}{2} t_w \phi^2 + \frac{1}{4} \phi^4 \right) dx \\ &= \frac{l}{4} \left[ \left( \frac{\pi^2}{l^2} - t_w \right) \phi_0^2 + \frac{3}{8} \phi_0^4 \right]. \end{aligned} \quad (18)$$

Letting  $dH/d\phi_0 = 0$  we get

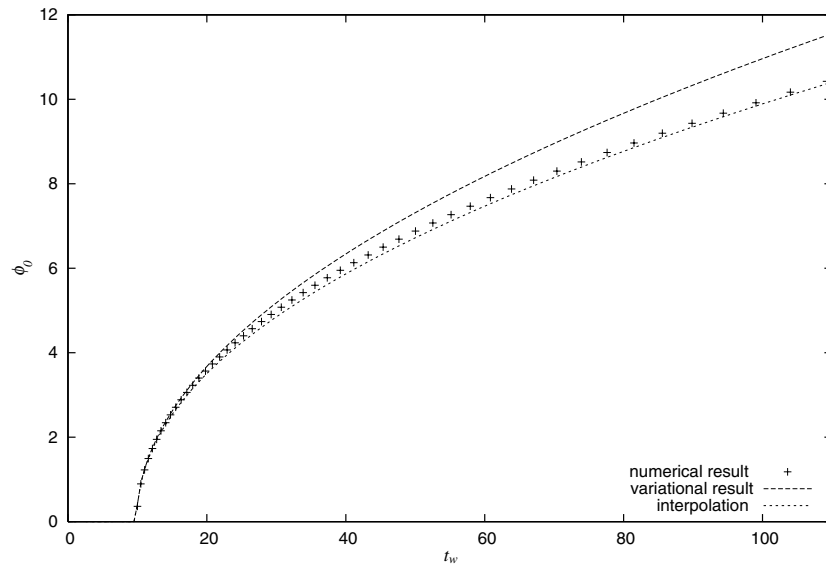
$$\phi_0 = 0 \quad \phi_0 = \pm \sqrt{\frac{4}{3} \left( t_w - \frac{\pi^2}{l^2} \right)}. \quad (19)$$

From this trial solution one see clearly how the well size depresses the amplitude  $\phi_0$ . The solution is nonzero only if  $t_w > \pi^2/l^2$ , which restores the criterion in the exact solution (equation (16)). The variational method works well for the cases with  $l \approx \pi/\sqrt{t_w}$  and hence  $\phi_0 \ll \sqrt{t_w}$ , because for  $\phi_0 \ll \sqrt{t_w}$ , the last term in equation (9) is not as important as the first two terms and then the triangle functions are indeed the solutions. As  $l \rightarrow \infty$ , the variational solution is  $\sqrt{4t_w/3}$ , while the correct one should be  $\sqrt{t_w}$ . There is about 16% error. For  $l \gg 1/\sqrt{t_w}$  or  $t_w \gg 1/l^2$ , the solution is  $\sqrt{t_w}$  in the large part of the well and only in the regions with width  $1/\sqrt{t_w}$  near the boundary decays to zero. Therefore, for  $l \gg 1/\sqrt{t_w}$  it is not proper to adopt the variational solution.

We also solve the above case numerically. The detail of the numerical method is given in appendix B. In the numerical calculation we choose the well size  $l = 1.0$ . The saddle point equation is solved for different  $t_w$ . Figure 1 shows the amplitude (the maximum) of the saddle point solution. The cases with other well size can be obtained from the transformation (6).

It can be seen that the amplitude approaches zero as  $t_w$  approaches  $\pi^2/l^2 \approx 9.869$ , and the variational solution is coincident with the numerical solution. The variational solution deviates from the numerical solution as  $t_w$  increases. In order to avoid the deviation, we modify the variational result and get an interpolation formula for the amplitude. As  $t_w \rightarrow \pi^2/l^2$  the formula gives the variational solution, and as  $t_w \rightarrow \infty$  the formula gives  $\phi_0 = \sqrt{t_w}$ . The interpolation formula is given by

$$\phi_0 = \sqrt{\frac{t_w - \pi^2/l^2}{1 - \pi/(4l\sqrt{t_w})}}. \quad (20)$$



**Figure 1.** The amplitudes of the numerical and variational solutions of the saddle point equation and the interpolation fitting. The well size is  $l = 1.0$ .

As shown in figure 1, the interpolation formula agrees well with the numerical solution.

Now consider the case with finite barriers,

$$t(x) = \begin{cases} -t_w & |x| < l/2 \\ t_b & |x| > l/2 \end{cases} \quad (21)$$

where  $t_b > 0$  and  $t_b \neq \infty$ . If so how does the solution change? The numerical solution with  $l = 1.0$ ,  $t_w = 10.0$  for some typical  $t_b$  is shown in figure 2. Obviously the saddle point solution will decay outside the well. For  $t_b \gg t_w, 1/l^2$ , the decaying length is  $1/\sqrt{t_b}$  [3]. This decaying length is much smaller than  $1/\sqrt{t_w}$  and  $l$ . In this case it is not bad to take the barriers as infinitely high. For  $t_b < t_w$  the decaying length is  $1/\sqrt{t_w}$ , i.e. in the barrier  $|x| > l/2$  the saddle point solution will decrease to near zero in the scale  $1/\sqrt{t_w}$ . The numerical calculation indeed verifies this qualitative picture.

In addition it should be noted that the saddle point solution with  $t_b = 0.4747$  shown in figure 2 has a long tail. The decay in the barrier is very slow for  $t_b \ll 1/l^2$ .

We solve the saddle point equation with the well (21) with different  $t_w$  and  $t_b$  numerically. The amplitude of the saddle point solution is shown in figure 3.

For finite barriers the saddle point solution is not zero at the boundary of the well and will extend outside the well to some distance. Therefore, we introduce an effective well size

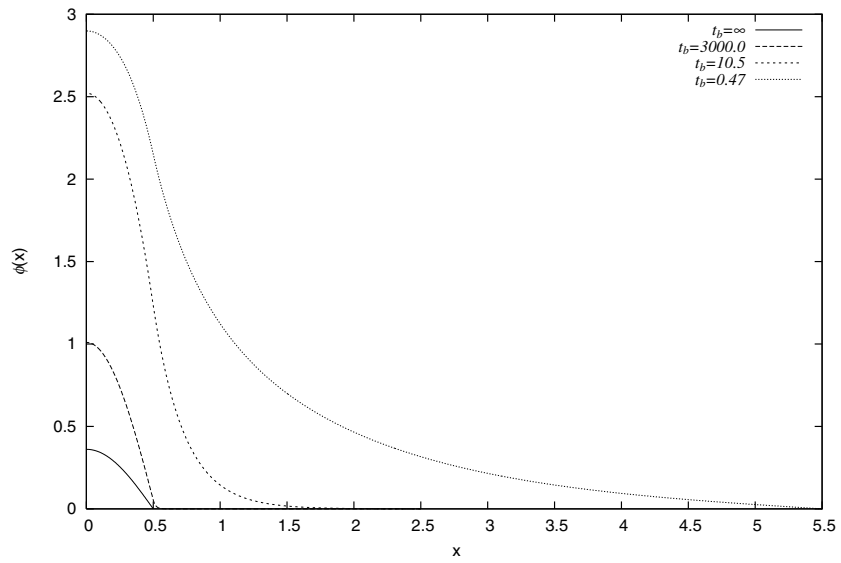
$$R_{\text{eff}} = l + \frac{c_1\sqrt{t_w} + c_2\sqrt{t_b}}{(\sqrt{t_w} + \sqrt{t_b})^2} \quad (22)$$

where  $c_1, c_2$  are two fitting parameters, to fit the result shown in figure 3 with

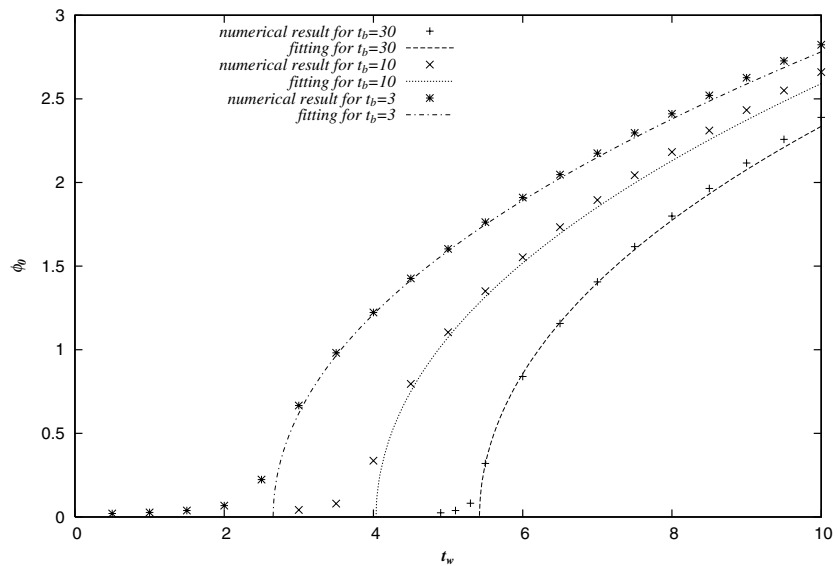
$$\phi_0 = \sqrt{\frac{t_w - \pi^2/R_{\text{eff}}^2}{1 - \pi/(4R_{\text{eff}}\sqrt{t_w})}}. \quad (23)$$

It is obtained from equation (20) by replacing  $l$  by  $R_{\text{eff}}$ . That is to say we try to describe the saddle point solution for the finite barrier with the trial solution (17) where  $l$  is replaced by





**Figure 2.** The solutions of the saddle point equation for the barrier height  $t_b = \infty, 3000, 10.5, 0.47$ . The well depth is  $t_w = 10.0$  and the well size is  $l = 1.0$ .



**Figure 3.** The amplitudes of the numerical solution of the saddle point equation and the fitting given by equation (23) with  $c_1 = 4.2, c_2 = 2.1$ . The well size is  $l = 1.0$ .

$R_{\text{eff}}$ . The fitting parameters in figure 3 are  $c_1 = 4.2, c_2 = 2.1$ . The rounding of the numerical results is likely because the precision is not enough. One can use this fitting formula to calculate the nonzero condition of the saddle point solution.

As one can see the fitting is good. Why can equation (23) take such a form? For  $\phi_0 \sim 0$ , the last term in equation (9) is not as important as the first two terms. If the size of the main part of the saddle point solution is  $R_{\text{eff}}$ , the differential term in equation (9) will lead

to  $\sim \phi_0 / R_{\text{eff}}^2$ . There are some reasons for us to adopt the form of equation (22) for  $R_{\text{eff}}$ . For  $t_b \gg t_w$  the decaying length in barriers is  $\sim 1/\sqrt{t_b}$ , and for  $t_b \ll t_w$  the decaying length in barriers is  $\sim 1/\sqrt{t_w}$  [3]. We make an interpolation between these two limits in equation (23). Therefore equation (23) works well for  $\phi_0 \sim 0$ .

From equation (23) one can see that Korzhenevskii's criterion [3] for the locally ordered cells, i.e.,  $t_w > 1/l^2$ , is too simplified. The threshold value of the well depth is only approximately proportional to the inverse of the square of the well size. More importantly, Korzhenevskii's criterion ignores the effects of the configuration of the cells. Whether the saddle point solution in a well is nonzero or not depends not only on itself but also on the surroundings. For different surroundings the criterion is different. The percolation of the locally ordered regions should not be the usual site percolation with Korzhenevskii's criterion.

In conclusion, for  $t_w, t_b \gg 1/l^2$ , the solution is  $\sqrt{t_w}$  in the large part of the well and only in the regions with width  $1/\sqrt{t_w}$  near the boundary decays to zero. For  $t_w, t_b \sim 1/l^2$ , the solution becomes very small and the threshold can be solved from equation (23).

The solution of a single well is helpful to understand the solution on a whole lattice. For  $\Delta \gg 1/l^2$ , the strongly disordered systems, the picture is clear. As  $t \sim 0$ , there are a lot of wells separated by barriers. Most probably, the depths of wells and heights of barriers satisfy  $|t_b|, |t_w| \gg 1/l^2$ . Therefore, there exist nonzero solutions in the wells. The tails of the localized solutions extending to the barriers are much shorter than the cell size, since the decaying lengths in barriers are  $\sim 1/\sqrt{t_b}$ . The saddle point solutions are well localized in single wells.

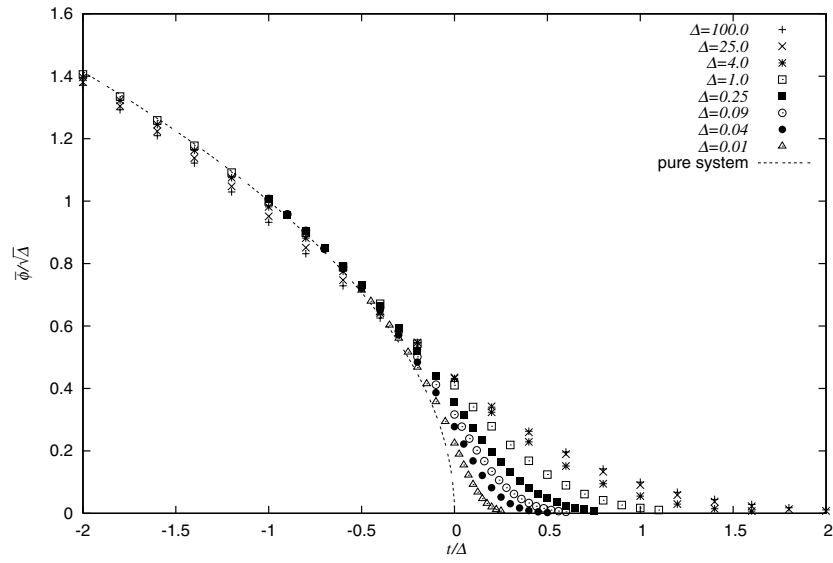
The calculation for the single well shows that whether there exists a nonzero solution depends on the depth of the well and the height of the barrier. For a lattice composed of wells and barriers one can apply the following mean-field argument. For  $t = 0$ , on average the well depths and the barrier heights equal the disorder strength  $\Delta$ . Taking  $t_b = t_w = \Delta$  in equation (23), one gets that for  $\Delta \approx 2.45/l^2$ , the solution will be zero in such a well. Naively one can claim that there exist localized saddle point solutions for  $\Delta > 2.45/l^2$ , while there exists no localized saddle point solution for  $\Delta < 2.45/l^2$ . However, it is wrong. In our numerical calculation on a lattice, it is found that even for  $\Delta \ll 1/l^2$  there still exist localized solutions. However the solutions are not localized within single wells. Because for  $\Delta \ll 1/l^2$ , as  $t \sim 0$ , the decaying length in the barriers  $\sim 1/\sqrt{\Delta} \gg l$ , the localized saddle point solutions extend over many cells, which can be wells and barriers. In the calculation for a single well the barriers extend to infinity. Therefore, the single well consideration discussed above is not suitable for the solution on a lattice for  $\Delta \ll 1/l^2$ . In fact  $\Delta > 2.45/l^2$  can only be an approximate condition that the saddle point solutions are localized in single wells.

#### 4. Numerical solutions on a lattice

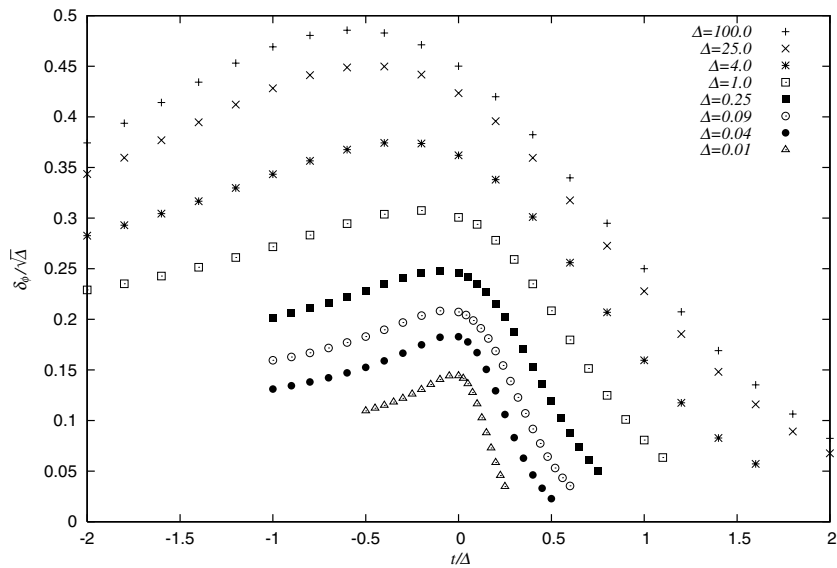
We have performed an extensive numerical calculation on a whole lattice with different disorder strengths  $\Delta$ . The size of the lattice cell is set to be  $l = 1$  and the coupling constant is set to  $g = 1$ . The details of numerical calculation are given in appendix B. In the calculation the lattices consist of  $10^4$ – $10^5$  cells. For a certain realization of  $\tilde{t}_i$ , we solve the saddle point equation. The periodic boundary condition is used. In this paper we only consider the ground state, in which there is no domain wall. We will discuss the excited states in the last section.

Since the solutions have randomness, we calculate the average of the solution

$$\bar{\phi} = \frac{1}{L} \int_0^L dx \phi(x) \quad (24)$$



**Figure 4.** The curves of  $\bar{\phi}/\sqrt{\Delta}$  versus  $t/\Delta$  of the saddle point solution for  $\Delta = 100, 25, 4, 1, 0.25, 0.09, 0.04, 0.01$ .



**Figure 5.** The curves of  $\delta_{\phi}/\sqrt{\Delta}$  versus  $t/\Delta$  of the saddle point solution for  $\Delta = 100, 25, 4, 1, 0.25, 0.09, 0.04, 0.01$ .

where  $L$  is the length of the system, and the fluctuation around the average

$$\delta_{\phi} = \sqrt{\frac{1}{L} \int_0^L dx (\phi(x) - \bar{\phi})^2}. \tag{25}$$

The results of  $\bar{\phi}$  and  $\delta_{\phi}$  for different  $\Delta$  are shown in figures 4 and 5, respectively. We scale the temperature  $t$  with  $\Delta$ ,  $\bar{\phi}$  and  $\delta_{\phi}$  with  $\sqrt{\Delta}$ .

For strongly disordered systems  $\Delta \gg 1$ , the results can be understood by the consideration of single wells. The probability of wells is  $\frac{1}{\sqrt{\pi\Delta}} \int_{-\infty}^{-t} \exp(-\tilde{t}^2/\Delta^2) d\tilde{t} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-t/\Delta} \exp(-x^2) dx$ . For  $t > 2\Delta$ , there is almost no well. Therefore the average of the solution becomes very small for  $t > 2\Delta$ . As  $t$  decreases, the number of wells increases. Most probably the heights of barriers are much larger than 1 since  $\Delta \gg 1$ . Equation (20) can be a good criterion on whether the solution in a well is zero or not. Denote the temperature of the  $i$ th well by  $t + \tilde{t}_i$ . If  $-(t + \tilde{t}_i) > \pi^2$ , the solution in this well can be nonzero. For  $-(t + \tilde{t}_i) \gg \pi^2$ , the saddle point solution in the large part of the well is equal to  $\sqrt{-t - \tilde{t}_i}$  approximately. Therefore, the average of the order parameter is approximately given by  $\frac{1}{\sqrt{\pi\Delta}} \int_{-\infty}^{-t-\pi^2} \sqrt{-(t + \tilde{t})} \exp(-\tilde{t}^2/\Delta^2) d\tilde{t} \approx \frac{\sqrt{\Delta}}{\sqrt{\pi}} \int_{-\infty}^{-t/\Delta} \sqrt{-(t/\Delta + x)} \exp(-x^2) dx$ . This approximation becomes exact as  $\Delta \rightarrow \infty$ . Therefore  $\bar{\phi}/\sqrt{\Delta}$  depends only on  $t/\Delta$  for  $\Delta \gg 1$ . The curves  $\bar{\phi}/\sqrt{\Delta}$  versus  $t/\Delta$  should have the same shape for  $\Delta \gg 1$ . So it is for  $\delta_\phi/\sqrt{\Delta}$  versus  $t/\Delta$ . The curves for  $\Delta = 100, 25$  in figures 4 and 5 are indeed almost coincident.

For  $\Delta < 1$ , i.e.  $\Delta = 0.25, 0.09, 0.04, 0.01$ , as shown in figure 4,  $\bar{\phi}/\sqrt{\Delta}$  approaches zero faster than for  $\Delta = 100.0, 25.0$ . For comparison the curve for the pure system  $\bar{\phi} = \sqrt{t}$  is also drawn in figure 4. As  $\Delta$  approaches zero the curves approach that for pure systems. This is a natural result.

For the weakly disordered systems,  $\Delta \ll 1$ , the picture is not clear if we consider only a single cell. As  $t \sim 0$ , there are indeed a lot of cells with negative temperature, but the depths are about  $\Delta \ll 1/l^2$ . According to equation (23) it seems that the solution in the wells should be zero. However, the barrier heights are  $\sim \Delta$ , and the decaying length  $\sim 1/\sqrt{\Delta}$  is larger than the cell size. Although there will be no solution localized in single wells (except exponentially rare cases), there exist localized solutions extending over many cells. As shown later in figure 8,  $\phi_1$  is shown as a typical example of the saddle point solution for  $\Delta = 0.01$ . One can see that the localized solution actually extends over many cells. Note that the cell size is 1.

In order to describe the saddle point solution more explicitly, we define an auto-correlation function of the saddle point solution,

$$c(y) = \frac{1}{L} \int_0^L dx (\phi(x) - \bar{\phi})(\phi(x+y) - \bar{\phi}). \tag{26}$$

As an example, we show the correlation functions of the saddle point solutions in figure 6 for  $\Delta = 1$  at different temperatures. For other values of  $\Delta$  the correlation functions are similar.

It can be seen that  $c(y) \sim \exp(-y^2/\xi_\phi^2)$ , so we define the correlation length by

$$c(\xi_\phi) = c(0) e^{-1}. \tag{27}$$

The result of correlation length is shown in figure 7. The significance of this correlation length is that on this length scale, the saddle point solution varies slowly.

For  $\Delta = 100, 25, 4$ , the correlation length  $\xi_\phi$  is about 1. For  $t > \Delta$  the saddle point solutions are composed of localized solutions in wells, which are far from each other. The correlation length  $\xi_\phi$  is nearly equal to the width of the localized solutions. Therefore  $\xi_\phi$  is about 1. For  $t < 0$ , many incipient locally ordered regions should be connected; however, the correlation length is still the cell size. Since  $\Delta \gg 1$ , for most wells the depth is much larger than 1. Consider two neighbouring wells with local temperatures  $t_1, t_2$ , and  $-t_1, -t_2 \gg 1$ . In the large part of the wells the solutions are  $\sqrt{-t_1}, \sqrt{-t_2}$ , respectively. The sizes of the crossover regions are about  $1/\sqrt{-t_1}, 1/\sqrt{-t_2}$ , respectively and much smaller than the cell size. The solutions in these two wells are approximately not correlated, since  $t_1, t_2$  are not correlated. Therefore the correlation length is still about the cell size.

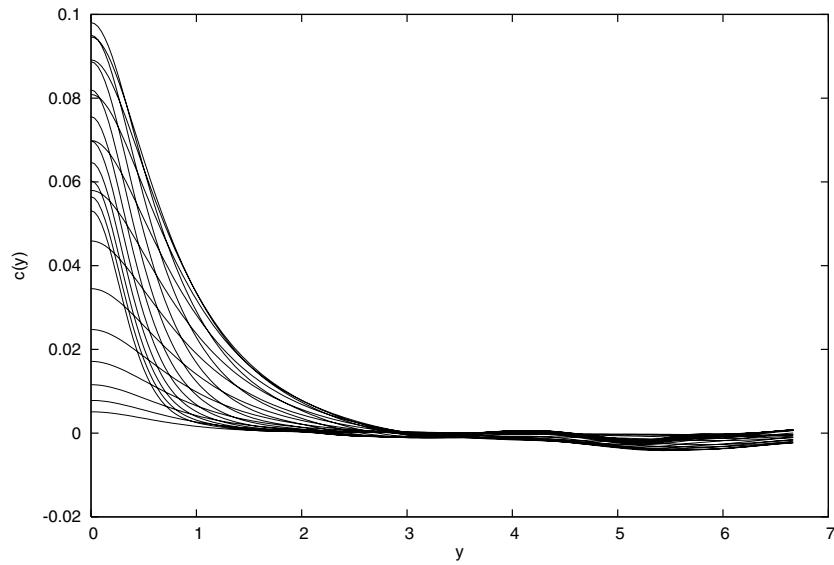


Figure 6. An example of the auto-correlation function at different temperatures  $t$  with  $\Delta = 1.0$ .

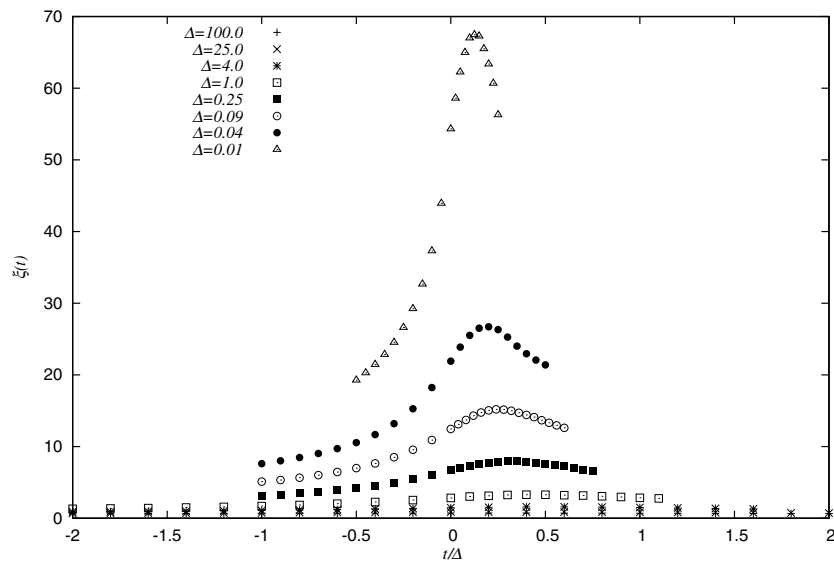
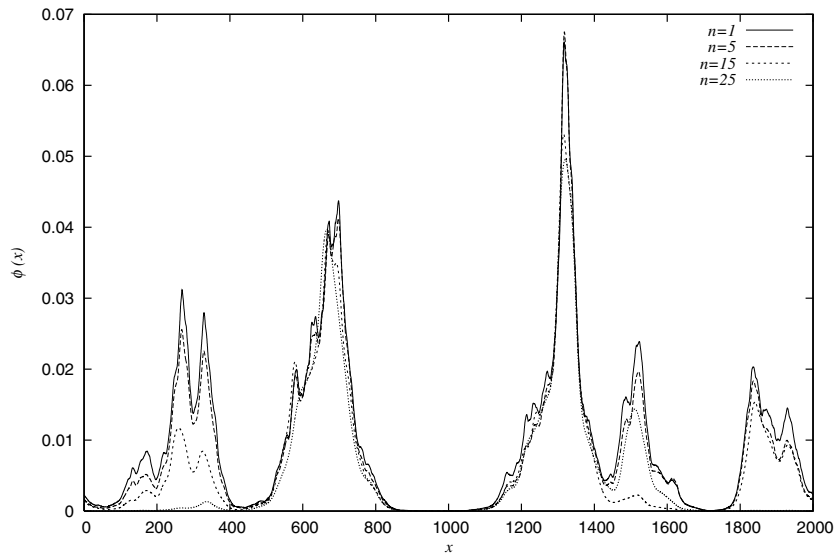


Figure 7. The correlation length  $\xi_\phi$  at different temperature  $t$  for  $\Delta = 100, 25, 4, 1, 0.25, 0.09, 0.04, 0.01$ .

For  $\Delta = 0.09, 0.04, 0.01$ , the correlation length  $\xi_\phi \gg 1$  as  $t \sim 0$ . As temperature decreases, the correlation length first increases, then after reaching a maximum, it decreases. One can understand this behaviour qualitatively as follows. On the side with large and positive  $t$ , incipient locally ordered regions are rare. Observing the saddle point solution directly one can see that it consists of many isolated peaks and these peaks extend over many cells (see figure 8). The significance of coarse-grained lattice in figure 8 is explained in the next section. The larger the temperature is, the smaller the heights and widths of the peaks are. Therefore



**Figure 8.** An example of the saddle point solutions on the coarse-grained lattices. The parameters for the solution are  $\Delta = 0.01$ ,  $t = 0.025\Delta$ ,  $L = 2000$  and  $h = 0.2$ .

the correlation length decreases as the temperature increases. On the side with negative temperature  $t$ , for  $-t \gg \Delta$ , the random part is only a small perturbation. The correlation function of the order parameter should be the same as that of random temperature, for which the correlation length is 1. With these two limits for the two sides, how the correlation length varies with the temperature can be understood qualitatively.

Furthermore we find that the results for  $\Delta \ll 1$  can be well explained by using the coarse-grained approximation, which is discussed in the next section.

### 5. Coarse-grained approximation

The saddle point solution depends on the configuration of  $\{\tilde{t}_i\}$ , i.e.

$$\phi = \phi(\{\tilde{t}_i\}, t, l). \tag{28}$$

We ignore the coupling constant  $g$ , since we can consider only the case  $g = 1$  due to the transformation (6)–(8). Generally the statistical quantities, such as  $\bar{\phi}$ ,  $\delta_\phi$ ,  $\xi_\phi$ , depend only on  $t$ ,  $l$ , and the strength of disorder  $\Delta$ . Then we have

$$\bar{\phi} = \phi(\Delta, t, l) \quad \delta_\phi = \delta_\phi(\Delta, t, l) \quad \xi_\phi = \xi_\phi(\Delta, t, l). \tag{29}$$

In section 3, we solve the saddle point solution for lattices of cells with cell size  $l = 1$  and the distribution width of  $\{\tilde{t}_i\}$  being  $\Delta$ . For  $\Delta \ll 1$  the localized saddle point solution extends over many cells. Therefore the variation of the saddle point solution in one cell is small. The average temperature over many cells determines the solution, while the details of the cells are unimportant. Now we define a new lattice by taking  $n$ -connected cells as a unit. In the new lattice, each cell has a local temperature  $t + \tilde{t}_j^{(n)}$ , where

$$\tilde{t}_j^{(n)} = \frac{1}{n} \sum_{i=(j-1)n+1}^{nj} \tilde{t}_i. \tag{30}$$

Of course,  $n$  is an integer larger than 1. The distribution function of  $\tilde{t}_j^{(n)}$  is given by

$$p(\tilde{t}_j^{(n)}) = \int \delta\left(\tilde{t}_j^{(n)} - \sum_{i=1}^n \tilde{t}_i\right) \prod_{i=1}^n p(\tilde{t}_i) d\tilde{t}_i = \frac{1}{\Delta^{(n)}\sqrt{\pi}} e^{-(\tilde{t}_j^{(n)}/\Delta^{(n)})^2} \quad (31)$$

where the distribution width of  $\{\tilde{t}_j^{(n)}\}$  is given by

$$\Delta^{(n)} = \Delta/\sqrt{n}. \quad (32)$$

Denote the solution for the new lattice  $\phi_n(\{\tilde{t}_j^{(n)}\}, t, n)$ . It should be noted that the cell size of the new lattice is  $n$ . Providing that  $n$  is not very large, we can assume that

$$\phi(\{\tilde{t}_i\}, t, 1) \approx \phi_n(\{\tilde{t}_j^{(n)}\}, t, n). \quad (33)$$

This is not a hand-waving argument, but a conclusion through directly observing the solution on the new lattice. Figure 8 shows an example of solutions  $\phi_n$  with  $n = 1, 5, 15, 25$ , respectively. The parameters for this example are  $\Delta = 0.01, t = 0.025\Delta, L = 2000, h = 0.2$ . When the coarse-grained approximation is applied the solution is made smoother. For  $n = 5, 15$  the approximation is rather good, but for  $n = 25$  it is not so good.

Correspondingly we have

$$\bar{\phi}(\Delta, t, 1) \approx \bar{\phi}_n(\Delta^{(n)}, t, n) \quad (34)$$

$$\delta_\phi(\Delta, t, 1) \approx \delta_{\phi_n}(\Delta^{(n)}, t, n) \quad (35)$$

and

$$\xi_\phi(\Delta, t, 1) \approx \xi_{\phi_n}(\Delta^{(n)}, t, n). \quad (36)$$

Obviously the larger  $n$  is, the worse the approximation is. What is the upper bound of  $n$ ? In fact the coarse-grained approximation has another significance. For  $t = 0$  the barrier height on the new lattice is about  $\Delta^{(n)}$ . From the discussion in section 3, we know that if the decaying length in the barriers on this new lattice is shorter than the cell size, i.e.

$$1/\sqrt{\Delta^{(n)}} \leq n \quad (37)$$

the saddle point solution will be localized in single wells. Then we get, if

$$n \geq \Delta^{-2/3} \quad (38)$$

the saddle point solution will be localized in single cells in this lattice. The larger  $n$  is, the better the localization condition is satisfied. However, the larger  $n$  is, the worse the coarse-grained approximation is. Therefore, it is plausible to take the above equation with ‘=’ as the upper bound of  $n$ . This value is not only the decaying length in the barriers but also the half width of the localized solution approximately. For  $n$  larger than this value, the structure of locally ordered regions will be smeared in the solution  $\phi_n$ .

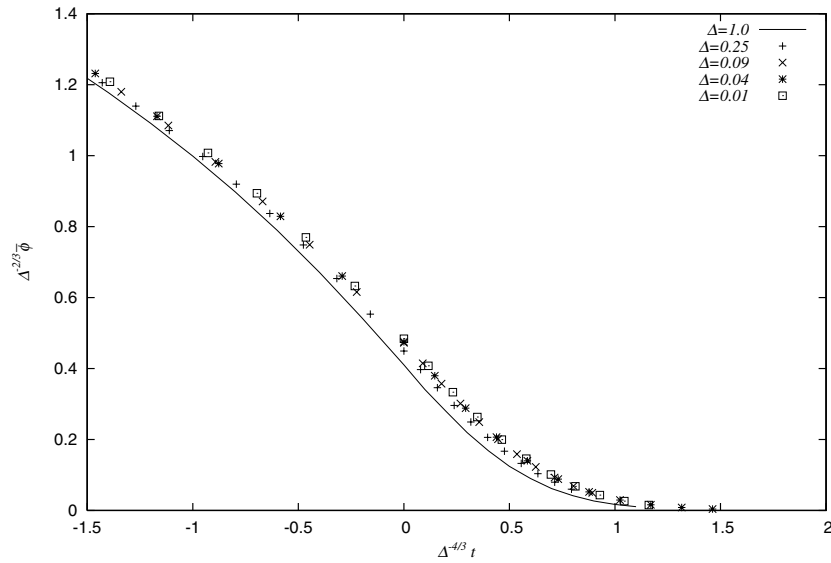
Since our calculation is carried out for a cell size of 1, we rescale the new lattice with transformation (6)–(8) to make the cell size 1. Applying transformation (6) with  $b = n$  to  $\phi_n(\{\tilde{t}_j^{(n)}\}, t, n)$ , we get

$$\phi_n(\{\tilde{t}_j^{(n)}\}, t, n) \equiv \frac{1}{n} \phi'_n(\{n^2 \tilde{t}_j^{(n)}\}, n^2 t, 1). \quad (39)$$

The average, fluctuation and correlation length of the transformed solution are given by

$$\bar{\phi}_n(\Delta^{(n)}, t, n) \equiv \frac{1}{n} \bar{\phi}'_n(n^2 \Delta^{(n)}, n^2 t, 1) \quad (40)$$

$$\delta_{\phi_n}(\Delta^{(n)}, t, n) \equiv \frac{1}{n} \delta_{\phi'_n}(n^2 \Delta^{(n)}, n^2 t, 1) \quad (41)$$



**Figure 9.** The rescaled results of  $\bar{\phi}$  with equation (43), for  $\Delta = 0.01, 0.04, 0.09, 0.25$ , respectively. The data are the same as in figure 4.

and

$$\xi_{\phi_n}(\Delta^{(n)}, t, n) \equiv n \xi_{\phi'_n}(n^2 \Delta^{(n)}, n^2 t, 1). \tag{42}$$

Letting  $n = \Delta^{-2/3}$ , which is the upper bound of  $n$ , and substituting it into equations (32), (34) and (40), we get

$$\bar{\phi}(\Delta, \Delta^{4/3} t, 1) \approx \Delta^{2/3} \bar{\phi}'_n(1, t, 1). \tag{43}$$

This equation says that the average of the solution for  $\Delta \ll 1$  can be mapped to that for  $\Delta = 1$  approximately. The result of this mapping is shown in figure 9. The data are the same as in figure 4.

Similarly we can get

$$\delta_{\phi}(\Delta, \Delta^{4/3} t, 1) \approx \Delta^{2/3} \delta_{\phi'_n}(1, t, 1). \tag{44}$$

The mapping is shown in figure 10. The data are the same as in figure 5. As we can see, the largest error is about 15% in this mapping. This is due to neglecting the detail of the temperature configuration under the scale  $n$ .

For the correlation length we have

$$\xi_{\phi}(\Delta, \Delta^{4/3} t, 1) \approx \Delta^{-2/3} \xi_{\phi'_n}(1, t, 1). \tag{45}$$

Figure 11 shows this mapping for  $\Delta = 0.25, 0.09, 0.04, 0.01$ . The data are the same as those in figure 7.

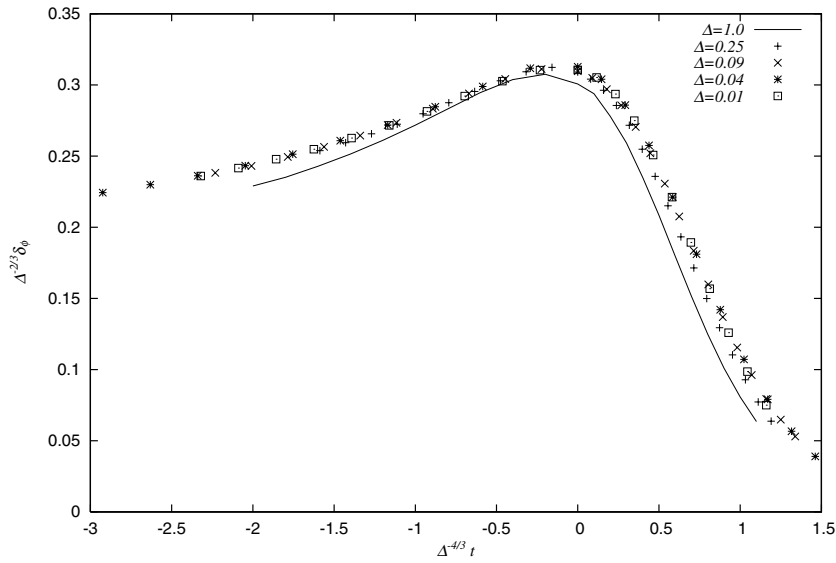
For  $t = 0$  it is

$$\xi_{\phi}(\Delta, 0, 1) \approx \xi_{\phi'_n}(1, 0, 1) \Delta^{-2/3}. \tag{46}$$

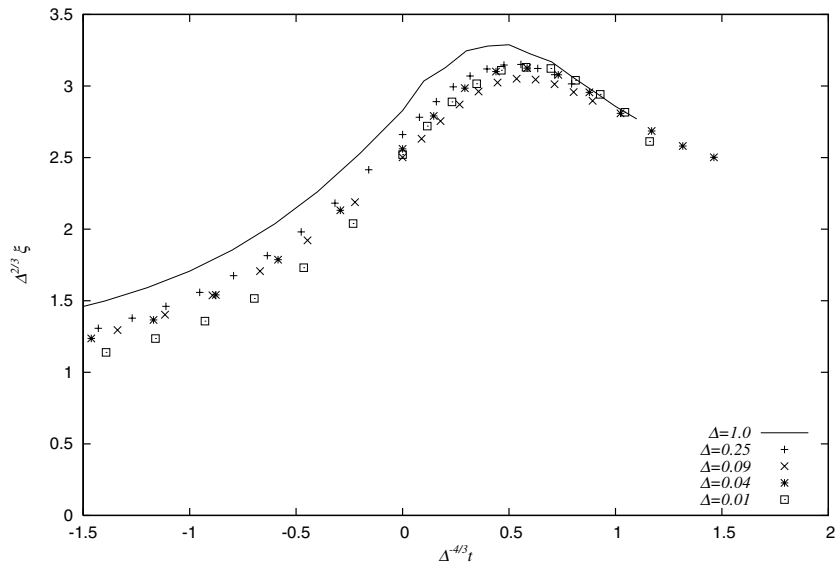
This is a power law. To verify it, we calculate the correlation length  $\xi_{\phi}$  of  $t = 0$  for different  $\Delta$  as shown in figure 10. For each  $\Delta$  we take the average over ten samples. For every sample the lattice sizes are  $8 \times 10^4$ . The fitting formula in figure 12 is given by

$$\ln \xi_{\phi} = a + b \ln \Delta \tag{47}$$





**Figure 10.** The rescaled results of  $\delta_\phi$  with equation (44), for  $\Delta = 0.01, 0.04, 0.09, 0.25$ , respectively. The data are the same as in figure 5.

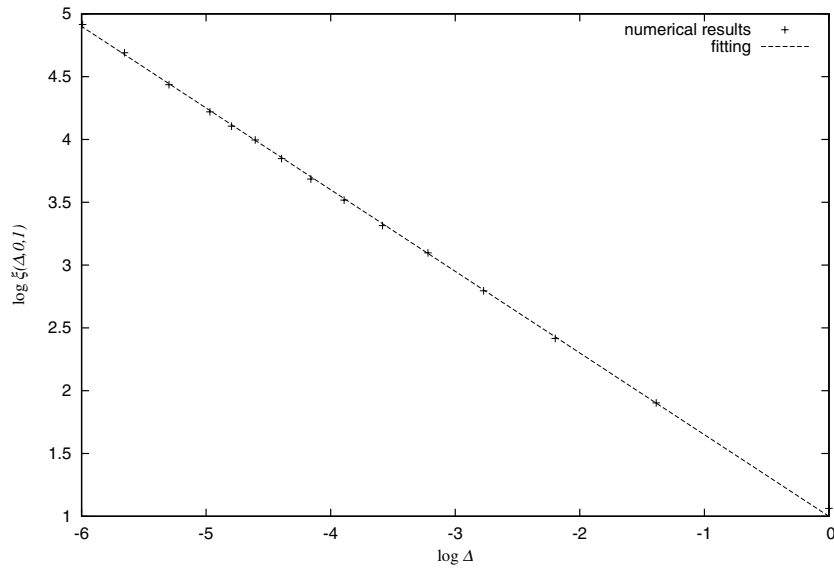


**Figure 11.** The rescaled results of  $\xi_\phi$  with equation (45), for  $\Delta = 0.01, 0.04, 0.09, 0.25$ , respectively. The data are the same as in figure 7.

where the fitting parameter  $a = 1.0, b = -0.65$ . That is to say

$$\xi_\phi(\Delta, 0, 1) = e^{1.0} \Delta^{-0.65}. \tag{48}$$

As one can see, the correlation length  $\xi \propto \Delta^{-0.65}$ . In the data, the relative deviation for different samples is about 3% for each  $\Delta$ . Because we take ten samples, the relative error should be about 1%. The coarse-grained approximation for the correlation length



**Figure 12.** The correlation length at  $t = 0$  for different  $\Delta$ . The fitting line is given by  $\ln \xi_\phi = 1.0 - 0.65 \ln \Delta$ .

agrees with the numerical solution surprisingly well. Comparatively, the coarse-grained approximation is worse for  $\bar{\phi}, \delta_\phi$  than that for  $\xi_\phi$  because  $\bar{\phi}, \delta_\phi$  depend on the detail of the random temperature more sensitively than  $\xi_\phi$ . From figure 8, one can see that when the coarse-grained approximation is applied, the solution is made smoother, while the size of the localization varies little.

The above argument can be easily generalized to  $d$ -dimensional systems. A new lattice with cell size  $n^d$  is defined. The distribution width of the random temperature on the new lattice is  $\Delta^{(n)} = n^{-d/2} \Delta$ . After the above procedures, one can get

$$\bar{\phi}(\Delta, \Delta^{4/(4-d)} t, 1) \approx \Delta^{2/(4-d)} \bar{\phi}'_n(1, t, 1) \tag{49}$$

$$\delta_\phi(\Delta, \Delta^{4/(4-d)} t, 1) \approx \Delta^{2/(4-d)} \delta_{\phi'_n}(1, t, 1) \tag{50}$$

and for correlation length we have

$$\xi_\phi(\Delta, \Delta^{4/(4-d)} t, 1) \approx \Delta^{-2/(4-d)} \xi_{\phi'_n}(1, t, 1). \tag{51}$$

For  $t = 0$  it is

$$\xi_\phi(\Delta, 0, 1) \approx \xi_{\phi'_n}(1, 0, 1) \Delta^{-2/(4-d)}. \tag{52}$$

From equation (52) one can see that for  $d > 4$  the correlation length  $\xi_\phi$  decreases as  $\Delta$  decreases. Then one get  $\xi_\phi < 1$  for  $\Delta < 1$ , which is unreasonable since the cell size is 1 and the correlation length cannot be much smaller than the cell size. One possibility is that the generalization to high dimension is wrong and there are still localized solutions for  $d > 4$  and  $\Delta < 1$ . Another one is that for  $d > 4$  there is no localized solution for  $\Delta < 1$ . It is because for  $d > 4$  the condition corresponding to equation (37) cannot be satisfied. If the latter possibility is valid, it can help us to understand Tarjus and Dotsenko’s spin-glass phase solutions of the disordered Ising ferromagnet [4], in which for  $d > 4$  there is no spin-glass solution.

Of course, the best way out of this dilemma is to solve the saddle point equation in high dimension.

## 6. Summary and discussion

We have solved the one-dimensional saddle point equation of the Ginzburg–Landau model with random temperature, which is correlated within the finite range. For strongly disordered systems,  $\Delta \gg 1/l^2$ , the incipient locally ordered regions are localized within single wells. The results can be understood by using the single well result. For weakly disordered systems,  $\Delta \ll 1/l^2$ , the incipient locally ordered regions extend over many cells. Since the average of the local temperature over many cells determines the saddle point solution, we propose a coarse-grained approximation, with which all the saddle point solutions for weakly disordered cases can be mapped to that for  $\Delta = 1$ .

Our results can give some enlightenment on the saddle point equation for high-dimensional systems. One can expect that the average, fluctuation and correlation of the saddle point solution are similar to the one-dimensional ones. More importantly, the solutions should be localized in single wells for  $\Delta \gg 1/l^2$  and extend over many cells for  $\Delta \ll 1/l^2$ . The generalization of the coarse-grained approximation to high dimension has been discussed.

We have solved only the ground state of the saddle point equation. For the excited states we can give a qualitative discussion. At  $t \gg \Delta$  the locally ordered regions are rare and far away from each other. Changing the signs of the locally order regions arbitrarily without changing the shape of the local solutions will lead to new solutions of the saddle point. Domain walls appear between the locally ordered regions. In these boundary regions the solution is exponentially small. The domain walls will cause an exponentially small increase of the energy. The average will be changed greatly. The fluctuation and the correlation function of the order parameter will be almost unchanged. At  $t \sim 0$  some incipient locally ordered regions are connected. The domain walls existing in the connected locally ordered regions will increase the energy. The competition between the ground state and the excited states will be very important for the phase transition. This is an important topic in the future study of the saddle point equation for high-dimensional systems.

In [3], for simplicity it is assumed that if two locally ordered regions are connected, the order parameters in these two regions have the same sign. Then the process of forming clusters of the incipient locally ordered regions with decreasing temperature becomes a pure percolation problem. In fact this scenario takes only the ground states into account for the connected locally ordered regions and ignores the excited states.

To solve the problem precisely, the excited states should be included. Only if the coupling of the two locally ordered regions is strong enough, will the order parameters in the two regions have the same sign in the thermodynamical sense. The energy gain by a domain wall depends not only on the saddle point solution, but also on the value of  $g$  in the Ginzburg–Landau Hamiltonian. Therefore, the process of forming clusters of the incipient locally ordered regions is not a simple percolation problem, if we still call this process a percolation.

Another important question is the type of percolation of the locally ordered regions. Tarjus and Dotsenko proposed that there may be a spin-glass phase in the random temperature Ising ferromagnet [4], where the random temperature is correlated with a  $\delta$ -function. However, there is a difficulty in their argument. To guarantee a spin-glass phase, it is necessary that the number of incipient spanning clusters should diverge with the size of the system, but in the usual site percolation it does not diverge with the size of the system for  $D < 6$  [15]. As shown in appendix A, the  $\delta$ -correlated disorder is a special case of finite-range correlated disorder with  $\Delta < 1$ . As discussed above the percolation of the ordered regions is not a simple and pure percolation. If in this kind of percolation the number of incipient spanning clusters diverges with the size of the system, the difficulty in Tarjus and Dotsenko's argument for the spin-glass phase might be solved. Therefore, the investigation of the percolation of the locally

ordered regions is important to the proposal of spin-glass phase in the random temperature Ising ferromagnet.

Finally, it should be pointed out that the saddle point equation (5) is similar to the nonlinear Schrödinger equation with randomness [17, 18], which is extensively studied on the wave propagation in nonlinear disordered media. The differences between the two equations are that the latter one is time dependent, and the term  $t\phi$  in the former one is replaced by the time partial differential  $i\partial\phi/\partial t$  in the latter one. The solution of the former one is a standing solution of the latter one. The comparison between them should be interesting.

**Acknowledgments**

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**Appendix A**

For the site-diluted classical spin models the Hamiltonian is given by

$$H = \frac{1}{2} \sum_{i,j} K_{i,j} p_i p_j \vec{S}_i \vec{S}_j \tag{53}$$

where  $\vec{S}_i^2 = m$ ,  $\vec{S}_i$  is a  $m$ -component spin variable,  $K_{ij}$  is a translationally invariant short-range ferromagnetic interaction and  $p_i$  is a random site variable having probability distribution

$$P(p_i) = p\delta(p_i - 1) + (1 - p)\delta p_i \tag{54}$$

where  $p$  is the site occupation probability and  $1 - p$  is the concentration of nonmagnetic impurities.

From this model, Grinstein and Luther have derived a replicated Hamiltonian [6]. To lowest order, it is given by

$$H(\{\phi_i^\alpha\}) = \frac{1}{2} \sum_{ij\alpha} L_{i,j} \phi_{i,\alpha} \phi_{j,\alpha} - pC_4 \sum_{j,\alpha} \phi_{j,\alpha}^4 + \frac{1}{2} V \sum_{ij\alpha\beta} \delta_{ij} \phi_{i,\alpha}^2 \phi_{j,\beta}^2 \tag{55}$$

where  $L_{ij} = (K^{-1})_{ij} - p\delta_{ij}$ ,  $V = (C_2)^2 p(1 - p)$  and  $C_2 = 1/2$ ,  $C_4$  is a constant. The replica index  $\alpha, \beta = 1, \dots, n$ ,  $n$  is set to be 0 at the end of the calculation.

Applying the continuation procedure,  $\sum_i \rightarrow a^{-d} \int d\mathbf{x}$ ,  $\delta_{ij} = a^d \delta(\mathbf{x} - \mathbf{y})$ , and letting  $a$  be the unit length, then one gets

$$H_n = \sum_{\alpha=1}^n \int d\mathbf{x} \left( \frac{1}{2} |\nabla \phi_\alpha(\mathbf{x})|^2 + \frac{1}{2} t \phi_\alpha^2(\mathbf{x}) + \frac{1}{4} g \phi_\alpha^4(\mathbf{x}) \right) - \frac{1}{8} p(1 - p) \sum_{\alpha,\beta=1}^n \int d\mathbf{x} \phi_\alpha^2(\mathbf{x}) \phi_\beta^2(\mathbf{x}). \tag{56}$$

On the other hand, using the usual replica trick to average the disorder [16], one can get the replicated Hamiltonian from (1) and (4)

$$H_n = \sum_{\alpha=1}^n \int d\mathbf{x} \left( \frac{1}{2} |\nabla \phi_\alpha(\mathbf{x})|^2 + \frac{1}{2} t \phi_\alpha^2(\mathbf{x}) + \frac{1}{4} g \phi_\alpha^4(\mathbf{x}) \right) - \frac{1}{8} \sum_{\alpha,\beta=1}^n \int d\mathbf{x} d\mathbf{y} c_t(\mathbf{x}, \mathbf{y}) \phi_\alpha^2(\mathbf{x}) \phi_\beta^2(\mathbf{y}). \tag{57}$$

Let the cell size in equation (4) be the lattice spacing, i.e.  $l = a$ . Provided that the order parameter varies slowly in the length scale of lattice spacing, which is a tacit assumption in getting the continuum representation from the discrete form, we can have

$$\int d\mathbf{x} d\mathbf{y} c_t(\mathbf{x}, \mathbf{y}) \phi_\alpha^2(\mathbf{x}) \phi_\beta^2(\mathbf{y}) \approx \frac{\Delta^2}{2} \int d\mathbf{x} \phi_\alpha^2(\mathbf{x}) \phi_\beta^2(\mathbf{x}) \quad (58)$$

where  $a$  is set to be the unit of length. Comparing equation (57) with (56), one get that for the diluted spin model the disorder in the effective Hamiltonian is correlated with the lattice spacing and the disorder strength is given by

$$\Delta = \sqrt{2p(1-p)}. \quad (59)$$

## Appendix B

In this appendix we give the numerical method to solve the saddle point equations. In one dimension the saddle point equation reads

$$-\frac{d^2\phi}{dx^2} + t(x)\phi + \phi^3 = 0. \quad (60)$$

We divide the  $x$ -axis into equally spaced segments, where the length of the segments is  $h$ . For the second derivative  $d^2\phi/dx^2$  we adopt the following difference formula [19]

$$\frac{d^2\phi}{dx^2} \rightarrow \frac{\phi_{-1} - 2\phi_0 + \phi_1}{h^2}. \quad (61)$$

Then we get the difference equation

$$-\frac{\phi_{i-1} - 2\phi_i + \phi_{i+1}}{h^2} + t_i\phi_i + \phi_i^3 = 0. \quad (62)$$

To solve this difference equation we use an iterative method

$$\phi_i^{(n+1)} = \omega\phi_i^{(n)} + 0.5(1-\omega)((\phi_{i-1}^{(n+1)} + \phi_{i+1}^{(n)}) - h^2(t_i\phi_i^{(n)} + \phi_i^{(n)3})) \quad (63)$$

where  $\phi_i^{(n)}$  is the value of the  $n$ th iteration, and  $\omega > 1$  is the over relaxation parameter. Choosing proper initial values and over relaxation parameter can enhance the calculation speed greatly.

In section 3, the well size is  $l = 1$  and the length of segments is  $h = 0.02$ . The systematic error of formula (61) is of the order  $h^4$  [19]. Therefore, the error due to the difference formula is of order  $10^{-7}$ . We end the iteration until  $\sum_i (\phi_i^{(n+1)} - \phi_i^{(n)})^2 / \sum_i (\phi_i^{(n)})^2 < 10^{-14}$ .

In section 4, for each  $\Delta$  we solve the equation for five samples. For each sample there are  $10^4$  cells for  $\Delta = 100, 25, 4, 1$ , and each cell is divided into 30 equally spaced segments. As discussed in section 2, we let the length of each cell be 1. Then the length of the segment is  $h = 1/30$ . For  $\Delta = 0.25, 0.09$ , for each sample there are  $4 \times 10^4$  cells and each cell is divided into ten segments. The length of each segment is 0.1. For  $\Delta = 0.04, 0.01$ , for each sample, there  $8 \times 10^4$  cells, and each cell is divided into five segments. The length of each segment is 0.2. Because for  $\Delta \ll 1$ , the saddle point solutions extend over many cells, the details of temperature in a single cell are not important. Therefore, we can take segments with large length without decreasing the precision.

Because the lattice size is very large,  $L \sim 10^4-10^5$ , the deviation from sample to sample is small. Therefore, we only take five samples for each set of parameters. For example, the relative deviation from the solution average is defined by  $\sqrt{\frac{1}{5} \sum_{I=1}^5 (\bar{\phi}_I - \bar{\phi}_{av})^2} / \bar{\phi}_{av}$ , where  $\bar{\phi}_{av} = \frac{1}{5} \sum_{I=1}^5 \bar{\phi}_I$  and  $\bar{\phi}_I$  is the average of the  $I$ th sample. For  $\Delta = 100, t = 0$ , this deviation

is 1.83%. It can be estimated in the following way. Obviously the deviation of averaged quantities from sample to sample is proportional to that of the numbers of localized peaks. For  $\Delta = 100$ ,  $t = 0$ , there are about 5000 wells in each sample. Suppose that there is a localized peak in each well, the deviation of the numbers of localized peaks from sample to sample is just  $1/\sqrt{5000} \approx 1.41\%$ , which agrees with the above numerical result 1.83%. For  $\Delta = 100$ ,  $t = 2\Delta$ , there are about  $10^4 \times \int_{-\infty}^{-2} \exp -x^2 dx \approx 23$  wells in each examples. Then the deviation is estimated to be 20%. The numerical result is 10%. For  $\Delta = 1$ , the deviation is 3.32% for  $t = 0$  and 12.4% for  $t = 1.1$ . For  $\Delta = 0.01$ , the deviation is 3.29% for  $t = 0$  and 18% for  $t = 0.0025$ . For all  $\Delta$ , the deviation for  $t < 0$  is smaller than that for  $t = 0$ . On the side of large  $t$  there are rare localized peaks, so the deviation is large. Therefore, the error in numerical calculation for the side of large  $t$  is large. However, the side of large  $t$  is far away from the percolative regime, and we are interested in  $t/\Delta \sim 0$  for  $\Delta > 1$ , and  $t/\Delta^{4/3} \sim 0$  for  $\Delta < 1$ , so the small number of samples will not cause serious problems. Of course, in order to verify the conclusion obtained in this paper more precisely, it is necessary to carry out the calculation with larger lattice size and more samples.

For the calculation in section 4, We end the iteration until  $\sum_i (\phi_i^{(n+1)} - \phi_i^{(n)})^2 / \sum_i (\phi_i^{(n)})^2 = \text{err}0 < 10^{-10}$ . For each parameter we testify the precision of calculation by lowering  $\text{err}0$  to  $10^{-11}$ ,  $10^{-12}$  or increasing the number of segments in each cell. We find that the relative deviation of the calculation result, such as  $\bar{\phi}$ , is less than  $10^{-3}$ . Since the relative deviation from different samples is about  $10^{-2}$  for  $t = 0$ , the precision of our calculation is enough.

## References

- [1] Dotsenko V S *et al* 1995 *J. Phys. A: Math. Gen.* **28** 3093
- [2] Dotsenko V S and Feldman D E 1995 *J. Phys. A: Math. Gen.* **28** 5183
- [3] Korzhenevskii A L, Heuer H-O and Herrmanns K 1998 *J. Phys. A: Math. Gen.* **31** 927
- [4] Tarjus G and Dotsenko V 2002 *J. Phys. A: Math. Gen.* **35** 1627
- [5] Lubensky T C 1975 *Phys. Rev. B* **11** 3537
- [6] Grinstein G and Luther A 1976 *Phys. Rev. B* **13** 1329
- [7] Weinrib A and Halperin B I 1983 *Phys. Rev. B* **27** 413
- [8] Bulaevskii L N, Panyukov S V and Sadovskii M V 1987 *Zh. Eksp. Teor. Fiz.* **92** 672  
Bulaevskii L N, Panyukov S V and Sadovskii M V 1987 *Sov. Phys.-JETP* **65** 380 (Engl. Transl.)
- [9] Pentegov V I and Feigel'man M V 1988 *Zh. Eksp. Teor. Fiz.* **94** 345  
Pentegov V I and Feigel'man M V 1988 *Sov. Phys.-JETP* **67** 2145 (Engl. Transl.)
- [10] Stauffer D and Aharony A 1992 *Introduction to Percolation Theory* 2nd edn (London: Taylor and Francis)
- [11] Plantevin O and Glyde H R 2002 *Phys. Rev. B* **65** 224505
- [12] Yoon J *et al* 1998 *Phys. Rev. Lett.* **80** 1461
- [13] Chan M H W *et al* 1988 *Phys. Rev. Lett.* **61** 1950  
Wong G K S *et al* 1990 *Phys. Rev. Lett.* **65** 2410  
Mulders N *et al* 1991 *Phys. Rev. Lett.* **67** 695
- [14] Schmidt H and Schwabl F 1978 *Z. Phys. B* **30** 197
- [15] Aizenman M 1996 *Preprint cond-mat/9609240*
- [16] Emery V J 1975 *Phys. Rev. B* **11** 239
- [17] Kivshar Yu S *et al* 1990 *Phys. Rev. Lett.* **64** 1693
- [18] Knapp R 1995 *Physica D* **85** 496
- [19] Koonin S E 1986 *Computational Physics* (New York: Benjamin/Cummings)